16.9 The Divergence Theorem

The Divergence Theorem

Recall: Green's Theorem in vector form

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where *C* is the **positively oriented boundary** curve of the plane region *D*.

Similarly, the **Divergence Theorem** gives the relationship between a **triple integral** over a solid region *E* and a **surface integral** over the surface of *E*.

Note, the surface *S* is **closed** in the sense that it forms the **complete boundary** of the solid *E*. Typical **examples** of **closed surfaces** are regions bounded by spheres, ellipsoids, cubes, tetrahedrons, or combinations of these surfaces.

The Divergence Theorem

The Divergence Theorem

Let ***E*** be a **solid region** bounded by a **closed surface *S*** oriented by a unit normal vector directed outward from *E*. If **F** is a vector field whose component functions have continuous first partial derivatives in *E*, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

The Divergence Theorem are for regions that are simultaneously of **types 1, 2, and 3** and call such regions **simple solid regions**.

Also, the Divergence Theorem states that, **under the given conditions**, the **flux of F** across the **boundary surface of *E*** is equal to the **triple integral** of the **divergence of F** over *E*.

Example – Verifying Divergence Theorem

Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + yz\,\mathbf{j} + 3z^2\,\mathbf{k}$ on the solid bounded by the **paraboloid** $z = x^2 + y^2$ and the plane $z = 1$.

Example – Using the Divergence Theorem

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ that is, calculate the flux of **F** across *S*.

$$\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} - x^2 z\,\mathbf{j} + z^2 y\,\mathbf{k},$$

S is the surface of the rectangular box bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$

Example – Using the Divergence Theorem

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ that is, calculate the flux of **F** across *S*.

$$\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k},$$

S is the sphere $x^2 + y^2 + z^2 = 1$

Example – Using the Divergence Theorem

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ that is, calculate the flux of **F** across *S*.

$$\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k},$$

S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

Example – Using the Divergence Theorem

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ that is, calculate the flux of **F** across *S*.

$$\mathbf{F}(x, y, z) = z \cos y \, \mathbf{i} + x \sin z \, \mathbf{j} + xz \, \mathbf{k},$$

S is the surface of the tetrahedron bounded by the planes $x = 0$ $y = 0$, $z = 0$, and $2x + y + z = 2$

Example – Using the Divergence Theorem

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ that is, calculate the flux of **F** across *S*.

$$\mathbf{F}(x, y, z) = xy^2\,\mathbf{i} + yz\,\mathbf{j} + zx^2\,\mathbf{k},$$

S is the surface of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$